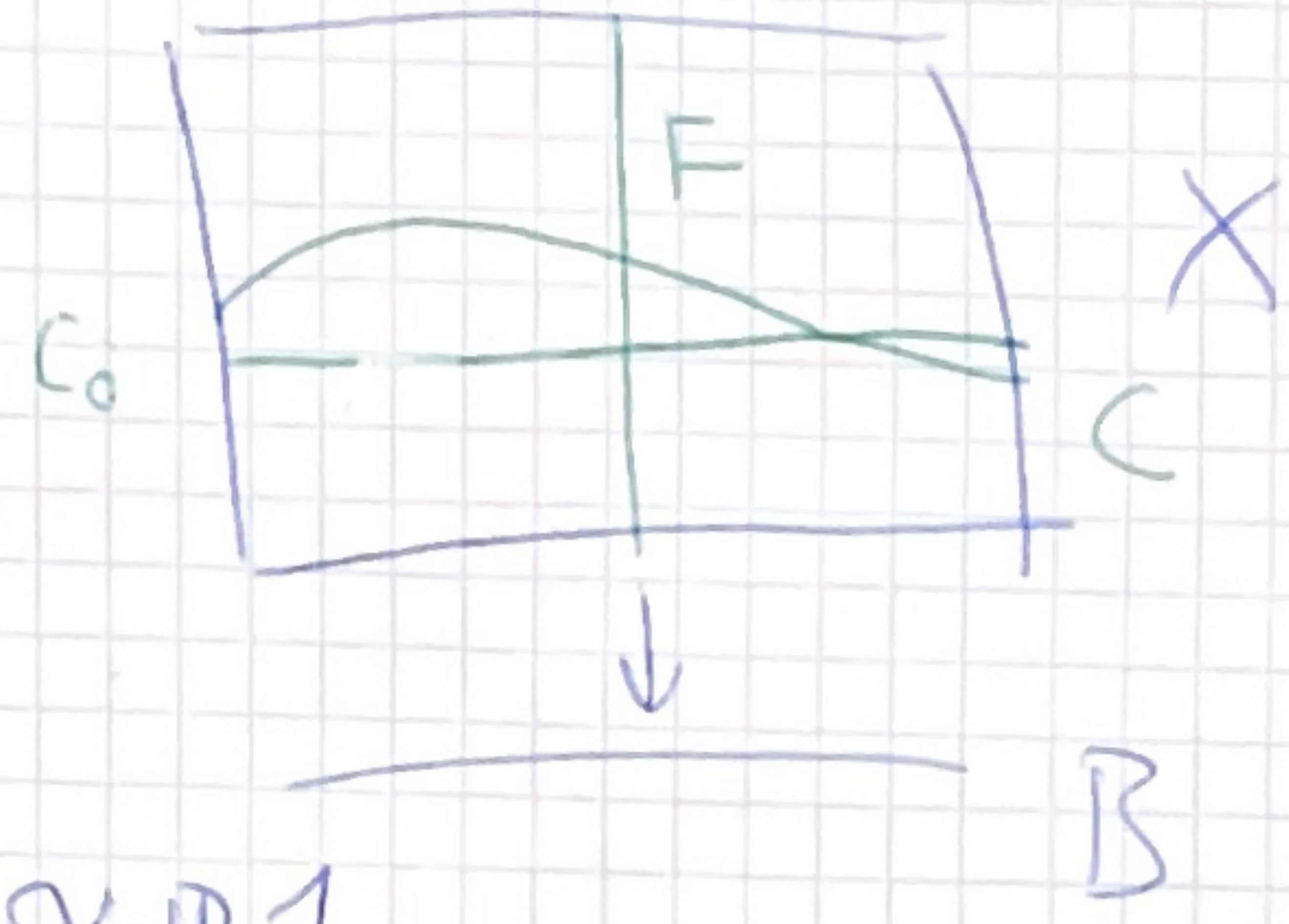
Ruled surfaces



Def ruled surface

$\pi : X \longrightarrow B$
surface — ns curve

s.t. 1) $\forall p \in B \quad X_p := \pi^{-1}(p) \cong \mathbb{P}^1$

2) $\exists \sigma : B \to X \quad \pi \circ \sigma = id_B$ section

Example $B \times \mathbb{P}^1 \longrightarrow B$

Lemma Any two fibres of a ruled surface are numerically equivalent.

Proof $F = X_p$, $F' = X_{p'}$ two fibres

By Bertini, suffices to show
$$(F.C) = (F'.C)$$
for all $C \overset{i}{\subseteq} X$ nonsingular irred.

The morphism $\pi \circ i : C \to B$ is finite or constant.

if $\pi \circ i$ is constant, then $X_{\pi(C)} = C$ is a fibre. $(C \hookrightarrow X_{\pi(C)} \cong \mathbb{P}^1)$

if $C \neq F, F'$, then $(F.C) = 0 = (F'.C)$

if $C = F$, then $(C.F - F') =$
$$(F.F) - (F.F') = (F.F)$$
$$= \deg_F(\mathcal{O}_X(F)|_F) = 0$$

$\mathcal{O}_X(F)|_F \cong (\pi \circ i)^* \mathcal{O}_B(p) \cong \mathcal{O}_F$

if $\pi \circ i$ finite, then

$$\deg_p(\mathcal{O}_X(F-F')|_C) = \deg((\pi\circ c)^* \mathcal{O}_B(p-p'))$$

$$= \deg(\pi\circ i) \cdot \underbrace{\deg(\mathcal{O}_B(p-p'))}_{=0} = 0$$

$\square$

<u>Lemma</u> 1) F a fibre of π

$D \in \mathrm{Div}(X)$ s.t. $(D.F) \geq 0$

Then $\pi_* \mathcal{O}_X(D)$ is locally free of rank $(D.F)+1$

2) $\pi_* \mathcal{O}_X \cong \mathcal{O}_B$

<u>Proof</u> 1) all fibres are $\mathbb{P}^1$'s

$\Rightarrow H^0(\mathcal{O}_X(D)|_{X_p}) = \deg(\mathcal{O}_X(D)|_{X_p}) + 1$

$= (D.X_p)+1 = (D.F)+1 \quad \forall p \in B$

& $H^1(\mathcal{O}_X(D)|_{X_p}) = 0 \quad \forall p \in B$

Cohomology and base change

$\Rightarrow \pi_* \mathcal{O}_X(D)$ is locally free of rank $(D.F)+1$.

2.) the adjunction map

$\mathcal{O}_B \to \pi_* \mathcal{O}_X$ is an isomorphism

indeed we have $\forall p \in B$

$\pi_* \mathcal{O}_X \otimes k(p) \xrightarrow{\cong} H^0(X_p, \mathcal{O}_{X_p})$ ②

$\Rightarrow \mathcal{O}_{B,p} \to \pi_* \mathcal{O}_{X,p}$ surjective $\underset{\text{cd. BC.}}{\Rightarrow}$ injective $\square$

Reminder on Projectivisations of vector bundles

S scheme, $\mathcal{E}$ locally free sheaf on S

$\mathbb{P}(\mathcal{E}) := \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{E}))\longrightarrow S$

no dual

is the projectivization of $\mathcal{E}$.

satisfies universal property.

$\mathrm{Hom}_{Sch/S}(\pi: X\to S, \mathbb{P}(\mathcal{E})\to S) = \{(\mathcal{L}, \pi^*\mathcal{E}\twoheadrightarrow\mathcal{L}) \mid \mathcal{L}\in\mathrm{Pic}(X/S)\}$

Thm

There is a canonical isomorphism

$\mathbb{P}(\mathcal{E})\cong\mathbb{P}(\mathcal{E}\otimes\mathcal{L})$

for every line bundle $\mathcal{L}$ on X.

Example let $\mathcal{E}$ be a locally free sheaf of rank 2 on B

Then $\mathbb{P}(\mathcal{E})\to B$ is a ruled surface.

Proof fibres $\cong\mathbb{P}^1$: $\mathbb{P}(\mathcal{E})\cong\mathbb{P}(k^{\oplus 2}) = \mathbb{P}^1$

existence of a section

let $U\subset B$ s.t. $\mathcal{E}|_U\cong\mathcal{O}_B^{\oplus 2}|_U$

Then $\mathbb{P}(\mathcal{E})|_U\cong\mathbb{P}(\mathcal{O}_U^{\oplus 2}) = U\times\mathbb{P}^1$ has a section.

$B\setminus U$ finite set of points

Use valuative criterion of properness to extend to $B\to\mathbb{P}(\mathcal{E})$

$\square$

Theorem B nonsingular curve

$$\left\{\begin{array}{l}\text{locally free}\\ \text{sheaves } \mathcal{E} \text{ of}\\ \text{rank 2 over } B\end{array}\right\} \Big/ \mathrm{Pic}(B) \longrightarrow \left\{\begin{array}{c}\text{fibred surfaces}\\ X \to B\end{array}\right\} \Big/ \underset{B}{\cong}$$

$$\mathcal{E} \longmapsto \mathbb{P}(\mathcal{E})$$

is a bijection

Proof $\pi : X \to B$ ruled surface.

$\sigma : B \to X$ section $C_\sigma = \sigma(B)$

$$(C_\sigma . F) = \deg_{C_\sigma}(\mathcal{O}_X(F)|_{C_\sigma}) \qquad F = X_p$$

$$= \deg_B(\sigma^* \pi^* \mathcal{O}_B(p)) = \deg_B(\mathcal{O}_B(p)) = 1.$$

$\overset{\text{lemma}}{\Longrightarrow}$ $\mathcal{E} := \pi_* \mathcal{O}_X(C_\sigma)$ rank 2 loc free sheaf.

claim that the adjunction map

$$\alpha : \pi^* \mathcal{E} = \pi^* \pi_* \mathcal{O}_X(C_\sigma) \to \mathcal{O}_X(C_\sigma)$$

is surjective

Indeed: NAK $\Rightarrow$ suffice to check α surjective on fibres.

$X_p \cong \mathbb{P}^1$, $\deg(\mathcal{O}_X(C_\sigma)|_{X_p}) = (C_\sigma . X_p) = 1$

$\Rightarrow \mathcal{O}_X(C_\sigma)|_{X_p}$ generated by global sections.

Cohomology and Base change

$\Rightarrow \mathcal{E} \otimes k(p) \twoheadrightarrow H^0(\mathcal{O}_X(C_\sigma)|_{X_p})$ is surjective.

$\Rightarrow \alpha$ surjective

By universal property of projectivisation, get $\varphi : X \to \mathbb{P}(\mathcal{E})$ over B s.t.

$$\varphi^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_X(C_\sigma)$$

$\Rightarrow X_p \hookrightarrow \mathbb{P}(\mathcal{E})_p$ closed embedding

$\Rightarrow \varphi : X_p \xrightarrow{\cong} \mathbb{P}(\mathcal{E})_p$ isomorphism on fibres

Zariski's Main Theorem $\Rightarrow$ φ isomorphism $\square$

<u>Corollary</u>. Ruled surfaces over B are birational to $B \times \mathbb{P}^1$

<u>Proof</u> $\mathbb{P}(\mathcal{E}) \to B$ ruled surface

$U \subset B$ open s.t. $\mathcal{E}|_U \cong \mathcal{O}_B^{\oplus 2}|_U$

$\Longrightarrow \mathbb{P}(\mathcal{E})|_U \cong \mathbb{P}(\mathcal{O}_B^{\oplus 2}) \cong \mathbb{P}^1 \times U$ $\square$

<u>Proposition</u>. $\pi : X \to B$ ruled surface

$C_\sigma = \sigma(B)$

$\Rightarrow \operatorname{Pic}(X) \cong \mathbb{Z} C_\sigma \oplus \operatorname{Pic}(B)$

$\operatorname{Num}(X) \cong \mathbb{Z} C_\sigma \oplus \mathbb{Z} F$

with $(C_\sigma . F) = 1$, $(F . F) = 0$

<u>Proof</u> $D \in \operatorname{Div}(X)$ $D' := D - (D.F) C_\sigma$

$\Rightarrow (D'.F) = (D.F) - (D.F)(C_\sigma . F) = 0$

Lemma $\Rightarrow \pi_* \mathcal{O}_X(D')$ invertible sheaf on B and $\mathcal{O}_X(D') \cong \pi^* \pi_* \mathcal{O}_X(D')$ (Coh. and BC)

$\Rightarrow \operatorname{Pic}(X)$ ~~is~~ generated by C_σ and $\pi^* \operatorname{Pic}(B)$

$\pi^*: Pic(B) \to Pic(X)$ has a section σ^*

$\sigma^* \circ \pi^* = id$

$\Rightarrow Pic(X) \cong \underbrace{Pic(X)/Pic(B)}_{\langle C_\sigma \rangle} \oplus Pic(B)$

$n(C_\sigma . F) = n \cdot \cancel{\ } \Rightarrow n\, C_\sigma \neq m\, C_\sigma \; n \neq m$ □

$$\boxed{\begin{array}{l}\underline{\text{Corollary}}: \; p_a(X) = g(B), \; p_g(X) = h^2(X, \mathcal{O}_X) = 0 \\ q(X) = h^1(X, \mathcal{O}_X) = g(B) \\ \underline{\text{Proof}} \; \pi_* \mathcal{O}_X = \mathcal{O}_B, \; h^1(\mathcal{O}_X) = h^1(\mathcal{O}_B), \; \chi(\mathcal{O}_X) = \chi(\mathcal{O}_B)\end{array}}$$

□

(Definition /)
Proposition: $\pi: X \to B$ ruled surface.

1) $\exists\, \mathcal{E}$ locally free rank 2 on B s.t.
$X \underset{B}{\cong} \mathbb{P}(\mathcal{E})$, $H^0(\mathcal{E}) \neq 0$, $\forall\, \mathcal{L} \in Pic(B)$, $\deg(\mathcal{L}) < 0$
$H^0(\mathcal{E} \otimes \mathcal{L}) = 0$
normalized

2) $e(X) = \deg(\mathcal{E}) = \deg(\Lambda^2 \mathcal{E})$ is an invariant of X/B.

3) $\exists\, \sigma_0 : B \to X$ s.t. $\mathcal{O}_X(\underbrace{\sigma_0(B)}_{= C_0}) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$

Proof $X \cong \mathbb{P}(\mathcal{E}')$ for some locally free $\mathcal{E}'$

for $\mathcal{M} \in Pic(B)$ with $\deg(\mathcal{M}) \gg 0$ have
$H^0(\mathcal{E}' \otimes \mathcal{M}) \neq 0$

The section $\sigma: B \to \mathbb{P}(\mathcal{E}')$ corresponds to
$0 \to \mathcal{N} \to \mathcal{E}' \to \mathcal{L} \to 0$ on B

$\deg(\mathcal{M}) \ll 0 \Rightarrow H^0(\mathcal{L} \otimes \mathcal{M}) = 0 \Rightarrow H^0(\mathcal{E}' \otimes \mathcal{M}) = 0$

Choose $\mathcal{M}$ s.t.

$$\deg(\mathcal{M}) = \min\left\{\deg(\mathcal{M}') \;\middle|\; \begin{array}{l}\mathcal{M}' \text{ s.t.} \\ H^0(\mathcal{E}'\otimes\mathcal{M}') \neq 0\end{array}\right\}$$

$\Rightarrow \mathcal{E} = \mathcal{E}'\otimes\mathcal{M}$ is normalised.

$\deg(\mathcal{E}) = \deg(\mathcal{E}') + \deg(\mathcal{M})$

$\Rightarrow$ ~~invariance follows~~

$e = \deg(\mathcal{E})$ invariant of X

to get a section as in 3):

pick $s \in H^0(\mathcal{E}) \setminus 0$ for ~~localised~~ normalised $\mathcal{E}$

have s.e.s.

$$0 \to \mathcal{O}_B \xrightarrow{s} \mathcal{E} \xrightarrow{q} \mathcal{L} \to 0$$

claim that $\mathcal{L}$ is a line bundle

suppose not.

$\mathcal{O}_C \subsetneq \mathcal{F} = \ker(\mathcal{E} \to \mathcal{L} \to \mathcal{L}^{tor} \to 0) \subseteq \mathcal{E}$

$\mathcal{F}$ torsion free of rank 1 on B

$\Rightarrow \mathcal{F}$ invertible with $\deg(\mathcal{F}) > 0$

$H^0(\mathcal{E}\otimes\mathcal{F}^\vee) \neq 0 \quad \Rightarrow \deg(\mathcal{F}^\vee) < 0$ ↯

$\Rightarrow q$ determines desired section.

<u>Proposition</u> $\pi : \mathbb{P}(\mathcal{E}) \to B$ / $\mathcal{E}$ normalised

$\mathcal{O}_X(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ with $\sigma_0 : B \to \mathbb{P}(\mathcal{E})$ corresponding to

$C_0 = \sigma_0(B)$

$\mathcal{E} \twoheadrightarrow \mathcal{L}$ corresp. to section $\sigma : B \to \mathbb{P}(\mathcal{E})$, $\sigma(B) = C_\sigma$

$\Rightarrow \deg_B(\mathcal{L}) = (C_\sigma . C_0)$ and $\mathcal{O}_X(C_\sigma)$

Lemma $\mathbb{P}(\mathcal{E}) \to B$ ruled surface
~~otella~~ $\mathcal{E} \twoheadrightarrow \mathcal{L}$ corresponding to
a section $\sigma : \cancel{\mathcal{E}}\, B \to \mathbb{P}(\mathcal{E})$
$\mathcal{N} := \ker(\mathcal{E} \twoheadrightarrow \mathcal{L})$ is a line
bundle and
$\mathcal{N} \cong \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-C_\sigma))$
$\pi^*\mathcal{N} \cong \cancel{\pi_*}\, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathcal{O}(-C_\sigma)$

Proof $\mathcal{N} \subsetneq \mathcal{E}$ rank 1 torsion free on B
$\Rightarrow$ line bundle.

$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}$

$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{C_\sigma} = \pi_* \underbrace{i_\sigma^*}_{=\sigma^*} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{L}$

so the s.e.s.
$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathcal{O}_X(-C_\sigma) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathcal{O}_{C_\sigma} \to 0$$
pushes down to
$$0 \to \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \mathcal{O}_X(-C_\sigma) \to \mathcal{E} \to \mathcal{L} \to 0$$

description of $\mathrm{Pic}(\mathbb{P}(\mathcal{E}))$
$+ \quad (F \cdot \mathcal{O}_X(1) \otimes \mathcal{O}_X(-C_\sigma)) =$
$= \deg_{\substack{F \\ \cong \mathbb{P}^1}} (\mathcal{O}_X(1) \otimes \mathcal{O}_X(-C_\sigma))|_F = 0$ $\square$

7.5 / 8.5

Proposition $\pi : \mathbb{P}(\mathcal{E}) \to B$, $\mathcal{E}$ normalised.

$\Rightarrow \forall\ \mathcal{E} \twoheadrightarrow \mathcal{L}$ corresp. to $\sigma : B \to \mathbb{P}(\mathcal{E})$

have $\deg_B(\mathcal{L}) = (C_\sigma . C_0)$

and $\mathcal{O}_X(C_\sigma) \cong \mathcal{O}_X(C_0) \otimes \pi^*(\mathcal{L} \otimes \det(\mathcal{E})^\vee)$

In particular $C_0^2 = -e$.

Proof $\mathcal{L} = \sigma_0^* \pi^* \mathcal{L} = \sigma_0^* \mathcal{O}_X(C_\sigma)$

$\Rightarrow \deg_B(\mathcal{L}) = \deg_{C_0}(\mathcal{O}_X(C_\sigma)|_{C_0}) = (C_0 . C_\sigma)$

s. es. $0 \to \mathcal{N} \to \mathcal{E} \to \mathcal{L} \to 0$

$\Rightarrow \mathcal{O}_X(C_0 - C_\sigma) \cong \pi^* \mathcal{N}$ $\square$